

1. The hyperbola  $H$  has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Find

(a) the coordinates of the foci of  $H$ , (3)

(b) the equations of the directrices of  $H$ . (2)

$$(a) \quad \frac{x^2}{16} - \frac{y^2}{9} = 1 \quad \Rightarrow \quad \begin{matrix} a=4 \\ b=3 \end{matrix}$$

Eccentricity:  $b^2 = a^2(e^2 - 1)$   
 $\therefore 9 = 16(e^2 - 1) \Rightarrow e = \frac{5}{4}$

$$\therefore \text{Foci} = (\pm 5, 0) \quad \text{(~~ae, 0~~)}$$

$$(b) \quad x = \pm \frac{a}{e} = \pm \frac{4}{5/4} = \pm \frac{16}{5}$$

$$\therefore x = \pm \frac{16}{5}$$

2.

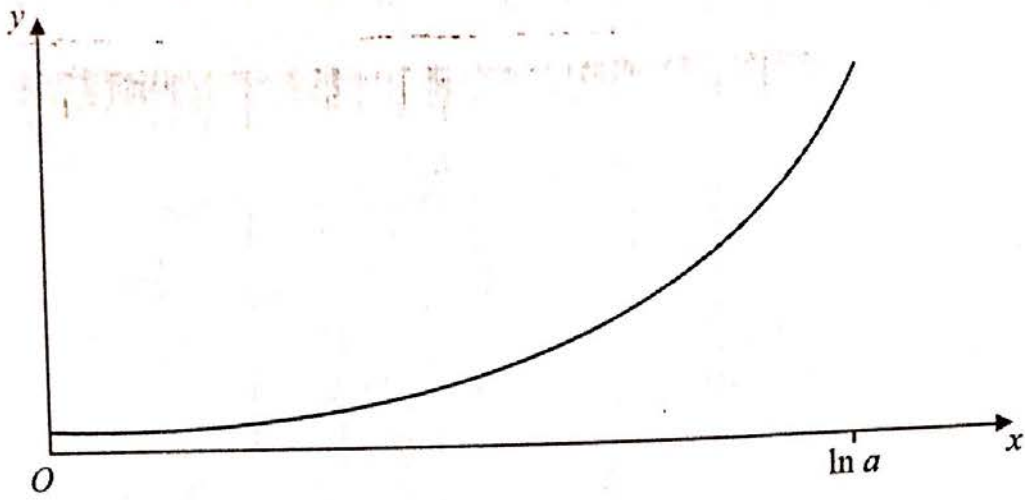


Figure 1

The curve  $C$ , shown in Figure 1, has equation

$$y = \frac{1}{3} \cosh 3x, \quad 0 \leq x \leq \ln a$$

where  $a$  is a constant and  $a > 1$

Using calculus, show that the length of curve  $C$  is

$$k\left(a^3 - \frac{1}{a^3}\right)$$

and state the value of the constant  $k$ .

(6)

2. Arc length =  $\int_0^{\ln a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

$$y = \frac{1}{3} \cosh 3x \Rightarrow \frac{dy}{dx} = \sinh 3x$$

$$c^2 - s^2 = 1$$

$$\cosh^2 = 1 + \sinh^2$$

$$\therefore \text{Length} = \int_0^{\ln a} \sqrt{1 + \sinh^2 3x} dx$$

$$= \int_0^{\ln a} \cosh 3x dx = \left[ \frac{1}{3} \sinh 3x \right]_0^{\ln a}$$



Question 2 continued

$$= \frac{1}{3} \sinh(3 \ln a) - \frac{1}{3} \sinh 0$$

$$= \frac{1}{3} \frac{e^{3 \ln a} - e^{-3 \ln a}}{2} - \frac{1}{3} \frac{e^0 - e^{-0}}{2}$$

$$= \frac{a^3 - a^{-3}}{6} - 0$$

~~$$= \frac{a^3 - a^{-3}}{6} \times \frac{a^3}{a^3}$$~~

$$= \frac{a^3 - a^{-3}}{6} = \frac{1}{6} \left( a^3 - \frac{1}{a^3} \right)$$

$$k = \frac{1}{6}$$

as  
required

(Total 6 marks)

Q2



3. The position vectors of the points  $A$ ,  $B$  and  $C$  relative to an origin  $O$  are  $i - 2j - 2k$ ,  $7i - 3k$  and  $4i + 4j$  respectively.

Find

(a)  $\overrightarrow{AC} \times \overrightarrow{BC}$ , (4)

(b) the area of triangle  $ABC$ , (2)

(c) an equation of the plane  $ABC$  in the form  $\mathbf{r} \cdot \mathbf{n} = p$  (2)

3(a).  $A = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$     $B = \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix}$     $C = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$

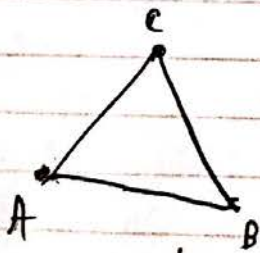
$\overrightarrow{AC} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix}$

$\therefore \overrightarrow{AC} \times \overrightarrow{BC} = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix} = \begin{vmatrix} 3 & 6 & 2 \\ -3 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 6 & 2 \\ -3 & 4 & 3 \end{vmatrix}$

$= \begin{pmatrix} 10 \\ -15 \\ 30 \end{pmatrix}$

(b)



Area =  $\frac{1}{2} |\overrightarrow{AC} \times \overrightarrow{BC}|$

$= \frac{1}{2} \left| \begin{pmatrix} 10 \\ -15 \\ 30 \end{pmatrix} \right| = \frac{35}{2} \text{ units}^2$



Question 3 continued

$$(C) \hat{A} = \begin{pmatrix} 10 \\ -15 \\ 30 \end{pmatrix}$$

$$\therefore \hat{L} \cdot \hat{A} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ -15 \\ 30 \end{pmatrix}$$

$$= 10 + 30 - 60 = -20$$

$$\therefore \hat{L} \cdot \begin{pmatrix} 10 \\ -15 \\ 30 \end{pmatrix} = -20$$



4.

$$I_n = \int_0^{\pi/4} x^n \sin 2x \, dx, \quad n \geq 0$$

(a) Prove that, for  $n \geq 2$ ,

$$I_n = \frac{1}{4} n \left( \frac{\pi}{4} \right)^{n-1} - \frac{1}{4} n(n-1) I_{n-2} \quad (5)$$

(b) Find the exact value of  $I_2$  (4)

(c) Show that  $I_4 = \frac{1}{64}(\pi^3 - 24\pi + 48)$  (2)

4(a). 
$$I_n = \int_0^{\pi/4} x^n \sin 2x \, dx$$

Let  $u = x^n \quad u' = n x^{n-1}$

$v' = \sin 2x \quad v = -\frac{1}{2} \cos 2x$

$$\therefore I_n = \left[ -\frac{x^n}{2} \cos 2x \right]_0^{\pi/4} + \frac{n}{2} \int_0^{\pi/4} x^{n-1} \cos 2x \, dx$$

$$\therefore I_n = \frac{n}{2} \int_0^{\pi/4} x^{n-1} \cos 2x \, dx$$

Let new  $u = x^{n-1} \quad u' = (n-1)x^{n-2}$

new  $v' = \cos 2x \quad v = \frac{1}{2} \sin 2x$

$$\therefore I_n = \frac{n}{2} \left( \left[ \frac{x^{n-1}}{2} \sin 2x \right]_0^{\pi/4} - \frac{(n-1)}{2} \int_0^{\pi/4} x^{n-2} \sin 2x \, dx \right)$$





Question 4 continued

$$\therefore I_n = \frac{n}{2} \left( \frac{(\pi/4)^{n-1}}{2} - \frac{(n-1)}{2} I_{n-2} \right)$$

$$\therefore I_n = \frac{n}{4} \left( \frac{\pi}{4} \right)^{n-1} - \frac{1}{4} n(n-1) I_{n-2}$$

as required.

$$(b) I_2 = \frac{1}{2} \left( \frac{\pi}{4} \right) - \frac{1}{2} I_0$$

$$= \frac{\pi}{8} - \frac{1}{2} \int_0^{\pi/4} \sin 2x \, dx$$

$$= \frac{\pi}{8} - \frac{1}{2} \left[ -\frac{1}{2} \cos 2x \right]_0^{\pi/4}$$

$$= \frac{\pi}{8} - \frac{1}{2} \left( 0 - -\frac{1}{2} \right)$$

$$= \frac{\pi}{8} - \frac{1}{4} = \frac{\pi-2}{8}$$

$$(c) I_4 = \left( \frac{\pi}{4} \right)^3 - 3 I_2 = \left( \frac{\pi}{4} \right)^3 - 3 \left( \frac{\pi-2}{8} \right)$$

$$= \frac{\pi^3}{64} - \frac{3\pi-6}{8} = \frac{\pi^3}{64} - \frac{8(3\pi-6)}{64}$$

$$= \frac{\pi^3}{64} - \frac{24\pi-48}{64} = \frac{1}{64} (\pi^3 - 24\pi + 48)$$

as required.



5. (a) Differentiate  $x \operatorname{arsinh} 2x$  with respect to  $x$ .

(b) Hence, or otherwise, find the exact value of

$$\int_0^{\sqrt{2}} \operatorname{arsinh} 2x \, dx$$

giving your answer in the form  $A \ln B + C$ , where  $A$ ,  $B$  and  $C$  are real. (7)

$$5(a). \quad \frac{d}{dx} (x \operatorname{arsinh} 2x) = \operatorname{arsinh} 2x + x \frac{d}{dx} (\operatorname{arsinh} 2x)$$

Consider  $\frac{d}{dx} (\operatorname{arsinh} 2x)$

$$\text{Let } y = \operatorname{arsinh} 2x$$

$$\sinh y = 2x$$

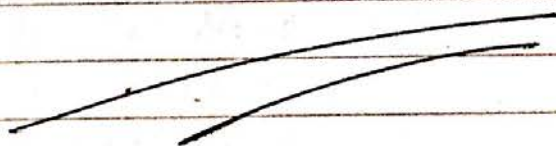
$$c^2 - s^2 = 1$$

$$\cosh y \frac{dy}{dx} = 2$$

$$\therefore \frac{dy}{dx} = \frac{2}{\cosh y} = \frac{2}{\sqrt{1 + \sinh^2 y}}$$

$$= \frac{2}{\sqrt{1 + 4x^2}}$$

$$\therefore \frac{d}{dx} (x \operatorname{arsinh} 2x) = \operatorname{arsinh} 2x + \frac{2x}{\sqrt{1 + 4x^2}}$$





$$(b) \frac{d}{dx} (x \operatorname{arsinh} 2x) = \operatorname{arsinh} 2x + \frac{2x}{\sqrt{1+4x^2}}$$

$$\therefore \operatorname{arsinh} 2x = \frac{d}{dx} (x \operatorname{arsinh} 2x) - \frac{2x}{\sqrt{1+4x^2}}$$

$$\therefore \int_0^{\sqrt{2}} \operatorname{arsinh} 2x \, dx = \int_0^{\sqrt{2}} \frac{d}{dx} (x \operatorname{arsinh} 2x) \, dx - \int_0^{\sqrt{2}} \frac{2x}{\sqrt{1+4x^2}} \, dx$$

$$= \left[ x \operatorname{arsinh} 2x \right]_0^{\sqrt{2}} - \frac{1}{4} \int_0^{\sqrt{2}} 8x (1+4x^2)^{-1/2} \, dx$$

$$= \sqrt{2} \operatorname{arsinh} 2\sqrt{2} - \frac{1}{4} \left[ 2(1+4x^2)^{1/2} \right]_0^{\sqrt{2}}$$

$$= \sqrt{2} \operatorname{arsinh} 2\sqrt{2} - \frac{1}{4} (6 - 2)$$

$$= \sqrt{2} \operatorname{arsinh} 2\sqrt{2} - 1$$

$$= \sqrt{2} \ln(2\sqrt{2} + \sqrt{9}) - 1$$

$$= \sqrt{2} \ln(3+2\sqrt{2}) - 1$$

$$A = \sqrt{2}$$

$$B = 3 + 2\sqrt{2}$$

$$C = -1$$



6. The ellipse  $E$  has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The line  $l_1$  is a tangent to  $E$  at the point  $P(a \cos \theta, b \sin \theta)$ .

(a) Using calculus, show that an equation for  $l_1$  is

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1 \quad (4)$$

The circle  $C$  has equation

$$x^2 + y^2 = a^2$$

The line  $l_2$  is a tangent to  $C$  at the point  $Q(a \cos \theta, a \sin \theta)$ .

(b) Find an equation for the line  $l_2$ . (2)

Given that  $l_1$  and  $l_2$  meet at the point  $R$ ,

(c) find, in terms of  $a$ ,  $b$  and  $\theta$ , the coordinates of  $R$ . (3)

(d) Find the locus of  $R$ , as  $\theta$  varies. (2)

$$6(a) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x = a \cos \theta \quad y = b \sin \theta$$

$$\frac{dy}{dx} = \frac{b \cos \theta}{-a \sin \theta} = -\frac{b}{a} \cot \theta$$

~~$$\therefore y - b \sin \theta = -\frac{b}{a} (x - a \cos \theta)$$~~

$$\therefore y - b \sin \theta = -\frac{b}{a} \cot \theta (x - a \cos \theta)$$

$$\therefore y - b \sin \theta = -\frac{b}{a} x \cot \theta + b \frac{\cos^2 \theta}{\sin \theta}$$



Question 6 continued

$$\therefore \textcircled{+ \sin \theta} \Rightarrow y \sin \theta - b \sin^2 \theta = -\frac{b}{a} x \cos \theta + b \cos^2 \theta$$

$$\textcircled{\div b} \Rightarrow \frac{y \sin \theta}{b} - \sin^2 \theta = -\frac{x \cos \theta}{a} + \cos^2 \theta$$

$$\therefore \frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = \cos^2 \theta + \sin^2 \theta$$

$$\therefore \frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$$

as required.

(b) Consider Circle:  $x = a \cos \theta$   $y = a \sin \theta$

$$\frac{dy}{dx} = \frac{a \cos \theta}{-a \sin \theta} = -\cot \theta$$

$$\therefore y - a \sin \theta = -\cot \theta (x - a \cos \theta)$$

$$\therefore y - a \sin \theta = -\frac{\cos \theta}{\sin \theta} x + \frac{a \cos^2 \theta}{\sin \theta}$$

$$\therefore y \sin \theta - a \sin^2 \theta = -x \cos \theta + a \cos^2 \theta$$

$$\therefore y \sin \theta + x \cos \theta = a (\cos^2 \theta + \sin^2 \theta)$$

$$\therefore y \sin \theta + x \cos \theta = a$$

as required.



Question 6 continued

$$(c) \quad \frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$$

$$\& \quad y \sin \theta + x \cos \theta = a$$

$$\Rightarrow \quad x \cos \theta = a - y \sin \theta$$

$$\therefore \quad \frac{a - y \sin \theta}{a} + \frac{y \sin \theta}{b} = 1$$

$\therefore$

$$\frac{ba - yb \sin \theta}{ab} + \frac{ay \sin \theta}{ab} = 1$$

$$\therefore \quad ba - yb \sin \theta + ya \sin \theta = ab$$

$$\therefore \quad \cancel{ba} + y \sin \theta (a - b) = ab$$

$$\therefore \quad y \sin \theta (a - b) = 0$$

$$\Rightarrow \quad y = 0$$

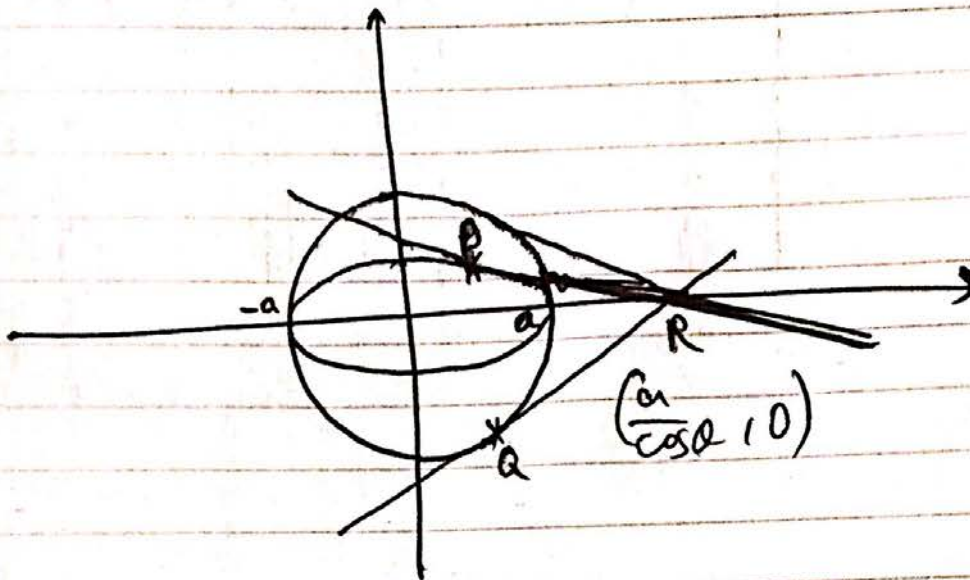
$$\Rightarrow \quad x \cos \theta = a \Rightarrow x = \frac{a}{\cos \theta}$$

$$\therefore \quad R = \left( \frac{a}{\cos \theta}, 0 \right)$$





(d)


 ~~$x_R$  cannot~~
 $x_R \notin [-a, a]$  because

otherwise tangents fall apart.

 $R$  lies on  $x$ -axis

 $\therefore$ 

$$x > a$$

~~$$x < -a$$~~

7.

$$f(x) = 5 \cosh x - 4 \sinh x, \quad x \in \mathbb{R}$$

(a) Show that  $f(x) = \frac{1}{2}(e^x + 9e^{-x})$

Hence

(b) solve  $f(x) = 5$

(4)

(c) show that  $\int_{\frac{1}{2} \ln 3}^{\ln 3} \frac{1}{5 \cosh x - 4 \sinh x} dx = \frac{\pi}{18}$

(5)

$$7(a). f(x) = 5 \cosh x - 4 \sinh x$$

$$= \frac{5e^x + 5e^{-x}}{2} - \frac{4e^x - 4e^{-x}}{2}$$

$$= \frac{5e^x + 5e^{-x} - 4e^x + 4e^{-x}}{2}$$

$$= \frac{e^x + 9e^{-x}}{2} = \frac{1}{2}(e^x + 9e^{-x})$$

as required.

(b)  $\frac{1}{2}(e^x + 9e^{-x}) = 5$

$$\therefore e^x + 9e^{-x} = 10$$

$$\therefore e^{2x} + 9 = 10e^x$$

$$\therefore e^{2x} - 10e^x + 9 = 0$$

$$(e^x - 9)(e^x - 1) = 0 \quad \begin{matrix} e^x = 9 \Rightarrow x = \ln 9 \\ e^x = 1 \Rightarrow x = 0 \end{matrix}$$

Scanned by CamScanner



Question 7 continued

$$(c) \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{\frac{1}{2}(e^n + 9e^{-n})} \, dn$$

$$= 2 \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{e^n + 9e^{-n}} \, dn$$

$$= 2 \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{e^n}{e^{2n} + 9} \, dn$$

~~$$= 2 \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2e^{2n}}{e^{2n} + 9} \, dn$$~~

Let  $u = e^n$

$$\frac{du}{dn} = e^n$$

$$\therefore dn = \frac{du}{e^n}$$

$$\therefore dn = \frac{du}{u}$$

~~$u = e^{\ln 3} = 3$~~   
 ~~$u = e^{\frac{1}{2}\ln 3}$~~

$$u = e^{\frac{1}{2}\ln 3} = \sqrt{3}$$

$$= 2 \int_{\sqrt{3}}^3 \frac{u}{u^2 + 9} \cdot \frac{1}{u} \, du$$

$$= 2 \int_{\sqrt{3}}^3 \frac{1}{u^2 + 9} \, du = 2 \left[ \frac{1}{3} \arctan \frac{u}{3} \right]_{\sqrt{3}}^3$$

Question 7 continued

$$= 2 \left( \frac{1}{3} \arctan 1 - \frac{1}{3} \arctan \frac{\sqrt{3}}{3} \right)$$

$$= 2 \left( \frac{\pi}{12} - \frac{\pi}{18} \right)$$

$$= 2 \times \frac{\pi}{36} = \frac{\pi}{18} \quad \text{as required.}$$



8. The matrix  $M$  is given by

$$M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

(a) Show that 4 is an eigenvalue of  $M$ , and find the other two eigenvalues. (5)

(b) For the eigenvalue 4, find a corresponding eigenvector. (3)

The straight line  $l_1$  is mapped onto the straight line  $l_2$  by the transformation represented by the matrix  $M$ .

The equation of  $l_1$  is  $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$ , where  $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$  and  $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ .

(c) Find a vector equation for the line  $l_2$ . (5)

8(a).  ~~$M =$~~

$$M - \lambda I = \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ -1 & 0 & 4-\lambda \end{pmatrix}$$

$$\therefore \det(M - \lambda I) = (2-\lambda)(2-\lambda)(4-\lambda) - (4-\lambda)$$

$$= (4-\lambda)((2-\lambda)^2 - 1)$$

$$= (4-\lambda)(\lambda^2 - 4\lambda + 3)$$

$$= (4-\lambda)(\lambda - 3)(\lambda - 1) = 0$$

$\therefore (4-\lambda) = 0 \Rightarrow \lambda = 4$  is indeed an e. value

$\lambda = 3$   $\lambda = 1$  are also eigenvalues.

$$(b) \quad Ax = 4x$$

$$\therefore \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2x + y \\ x + 2y \\ -x + 4z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

$$\therefore 2x + y = 4x \Rightarrow 2x = y$$

$$x + 2y = 4y \Rightarrow 2y = x$$

$$-x + 4z = 4z \Rightarrow x = 0$$

$$x = 0 \Rightarrow y = 0$$

$$\text{let } z = 1$$

$$\therefore \text{An e-vector is } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(c)



$$(c) \text{ eqn of } l_1 \quad (\underline{r} - \underline{a}) \times \underline{b} = 0$$

$$\therefore \underline{r} \times \underline{b} = \underline{a} \times \underline{b}$$

$$\therefore \underline{r} = \underline{a} + \lambda \underline{b}$$

$$\underline{r} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} 3 + \lambda \\ 2 - \lambda \\ -2 + 2\lambda \end{pmatrix}$$

$$M_{l_1} = l_2$$

$$\therefore \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 3 + \lambda \\ 2 - \lambda \\ -2 + 2\lambda \end{pmatrix} =$$

$$= \begin{pmatrix} 6 + 2\lambda + 2 - \lambda \\ 3 + \lambda + 4 - 2\lambda \\ -3 - \lambda - 8 + 8\lambda \end{pmatrix} = \begin{pmatrix} 8 + \lambda \\ 7 - \lambda \\ -11 + 7\lambda \end{pmatrix}$$

$$\therefore \underline{r} = \begin{pmatrix} 8 \\ 7 \\ -11 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$$

$$\therefore \left( \underline{r} - \begin{pmatrix} 8 \\ 7 \\ -11 \end{pmatrix} \right) \times \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = 0$$

